Concepts in Mathematics

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Chapter 2) What is Mathematics, is it a Science, and What are its Fundamental Components

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Definitions of Mathematics, and Related Concepts

Conventional Definitions of Mathematics

A very simplified definition of mathematics is a set of techniques based on numbers, for counting, and calculating quantities. This definition might represent the way most people experience mathematics, but it is incomplete. Mathematics certainly includes the above, but the term represents a much wider concept to mathematicians.

Presented below there are **six** definitions of mathematics from a number of sources. It is interesting to note that these definitions are worded differently, and it might seem that they are not referring to the same concept. However, they are all defining mathematics, from different perspectives. It may be insightful to access the original source, for additional information about these definitions, which can be done by clicking on the blue underlined words.

1) From the Oxford English Dictionary, 1933, Retrieved from wikipedia:

> "The abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations, and which includes as its main divisions geometry, arithmetic, and algebra"

2) From Weisstein, Eric W. "Mathematics. *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Mathematics.html

> "Mathematics is a broad-ranging field of study in which the properties and interactions of idealized objects are examined. Whereas mathematics began merely as a calculational tool for computation and tabulation of quantities, it has blossomed into an extremely rich and diverse set of tools, terminologies, and approaches which range from the purely abstract to the utilitarian."

3) From *Dictionary.com Unabridged, website*: http://dictionary.reference.com/browse/mathematics

> Mathematics "…the systematic treatment of magnitude, relationships between figures and forms, and relations between quantities expressed symbolically."

4) *From the Collins English Dictionary - Complete & Unabridged 10th Edition*. Retrieved from Dictionary.com website:

> "1.… a group of related sciences, including algebra, geometry, and calculus, concerned with the study of number, quantity, shape, and space and their interrelationships by using a specialized notation"

"2.… mathematical operations and processes involved in the solution of a problem or study of some scientific field"

5) *From the American Heritage® Science Dictionary*. Retrieved from Dictionary.com website:

"The study of the measurement, relationships, and properties of quantities and sets, using numbers and symbols. Arithmetic, algebra, geometry, and calculus are branches of mathematics."

6) *From the American Heritage® New Dictionary of Cultural Literacy, Third Edition*, Retrieved from Dictionary.com website:

"The study of numbers, equations, functions, and geometric shapes (*see* geometry) and their relationships. Some branches of mathematics are characterized by use of strict proofs based on axioms. Some of its major subdivisions are arithmetic, algebra, geometry, and calculus."

The above definitions are far from perfect. This is partly because of the limited number of words that are commonly used in dictionary definitions. In the following subsection, I attempt to create a definition that is more precise. To do this I started with a single paragraph of about 56 words defining mathematics. This was followed by five short paragraphs, of about 278 words, which define and explain the keywords that were used in the definition.

A Detailed Descriptive Definition of Mathematics

Based on the way I am using the terminology, mathematics is a **methodology based on logic**, and it consists of a set of **techniques** for counting, calculating quantities, and for carrying out **logical *computations*.** The *computations* can involve **formulas**, algorithms, symbols, geometric forms, as well as proofs based on deductive reasoning, involving **definitions**, **postulates**, and **theorems**.

The words in red type are defined and explained, based on the way they were used in this definition, in the following five paragraphs.

Formulas are statements usually written in the form of equations that indicate quantitative relationships between entities, and/or a set of instructions, for people or computers, on how to calculate the quantity of a specific entity, such as the following examples: $E = mc^2$ and $K = \frac{mv}{2}$.

In this definition, the word **techniques,** means any method, set of steps, or algorithm used by humans or computers for logical or quantitative computations. This includes the techniques used to solve algebraic equations, and to carry out arithmetic operations, such as addition, multiplication, subtraction, and division. This also includes the techniques and concepts from calculus, number theory, set theory, statistics, topology and from related subjects that involve logical relationships that are based on numbers, geometric forms, pathways, sequences, and/or symbols.

The words **logical computations**, refers to any type of reasoning carried out by human beings, and/or computers. This includes inductive and deductive reasoning, as well as experimentation. **However, mathematics is primarily based on deductive reasoning, involving formal proofs**.

The word **proofs** in this definition means a statement or argument that is based on deductive reasoning, which can result in a general theorem that may have wide applications. The arguments are comprised of definitions, postulates, and theorems.

The term **definitions** refers to the meaning of the concepts used in mathematics, such as numbers, lines, points, triangles, squares, circles, etc. The word **postulates**, means **axioms**, which are concepts that are assumed to be true, or are obviously true. Postulates are ultimately based on definitions, and the validity of a postulate can usually be determined by experimentation. **Theorems** in this definition mean a concept that can be proved with one or more postulates, and/or definitions, using deductive reasoning.

Mathematical Problems, and the Close Relationship of Logic, and Computer-Based Problems

Obvious examples of mathematics are addition subtraction multiplication, division, and solving equations and inequalities for unknowns. Proving theorems is another example of

mathematics. Defining sets, and calculating the Union of sets, Intersection of sets, and the Difference of sets are additional examples of mathematics. There are sometimes no numbers involved with mathematical computations. This is seen when proving most theorems, and in the following simple example of the intersection of two sets:

$$\{A, B, C, D, E\} \cap \{C, D, E, F, G, H\} = \{C, D, E\}$$

There is a very close relationship between mathematics and pure logic, especially symbolic logic. There is also a close relationship between mathematics, and computer code, and electronic computations. Often, there is an overlap between all of the above, and it can sometimes be difficult to classify a problem as mathematical, logical, or computer-based.

At the simplest level, logical problems involve computations to determine if a statement is true or false. Computers can carry out various types of computations, to determine if a statement is true or false. With computers, often true is represented by 1, (one) and false is represented by 0 (zero).

The following examples are problems that can be defined as mathematical, logical, or computer-based. These problems were solved with Microsoft Excel, which involve true or false computations. Excel will **display a number** when TRUE and/or FALSE is directly involved with addition, multiplication, division, or subtraction, such as TRUE*1=1, and FALSE*1=0. (The above represents multiplication by one, in a Microsoft Excel worksheet.)

*If any of the following statements are inserted into a **Cell** in Microsoft Excel, with the copy and paste function, the results presented in red type, will be displayed. If you want to download the original worksheet use for these calculations, left click on one of the following links: www.TechForText.com/Ma/L.xlsx* If you want to download the above in a zipped folder left click on the following: *www.TechForText.com/Ma/L.zip*

=(TRUE*1) Excel displays **1**

=(FALSE*1) Excel displays **0**

=(TRUE+TRUE+TRUE) Excel displays **3**

=(TRUE+TRUE+TRUE-5) Excel displays **-2**

=(2+4=6) Excel displays **TRUE**, because the statement is true

=(2+4=5) Excel displays **FALSE**, because the statement is false

=(2+4=6)+(4*4=16) Excel displays **2**, because both statements are true

=(TRUE*1&FALSE*1) Excel displays **10, because** TRUE*1=1, FALSE*1=0, and & means to combine both terms into 10

=(TRUE*1&FALSE*1+9) Excel displays **19, because** TRUE*1&FALSE*1=10, and 10+9=19

=(TRUE*1&FALSE*1&FALSE*1) Excel displays **100, because** TRUE*1=1, FALSE*1=0, FALSE*1=0, and the two &, means to combine the turns into 100

=(SIN(2.4)*SIN(2.4)+COS(2.4)*COS(2.4)=1) Excel displays **TRUE**, because the statement is true, based on the Pythagorean trigonometric identity: $(\sin Q)^2 + (\cos Q)^2 = 1$

Note, Microsoft Excel has logical functions that are more complex than the above. I used the TRUE / FALSE functionality, because it was the simplest for demonstration purposes.

Is Mathematics a Science, and What are the Components that Comprise Mathematics

Is Mathematics a Science?

Many sources call mathematics a science, and many people believe that mathematics is a property of nature. This can be seen in some of the definitions presented at the beginning of this chapter. Mathematics is actually a human invention, primarily based on logical proofs. However, it can be **superimposed** onto nature, and formulas can be created to represent qualitative and quantitative aspects of natural phenomena, such as gravity.

Some people would argue whether or not mathematics is a science depends on the definitions you are using. However, mathematics should **not** be defined as a science, **because important differences between science and mathematics will be concealed.** This will become apparent after reading the following definitions and explanations.

From the American Heritage® Science Dictionary, retrieved from Dictionary.com website:

> **Science** "The investigation of natural phenomena through observation, theoretical explanation, and experimentation, or the knowledge produced by such investigation. Science makes use of the scientific method, which includes the careful observation of natural phenomena, the formulation of a hypothesis, the conducting of one or more experiments to test the hypothesis, and the drawing of a conclusion that confirms or modifies the hypothesis.

My definition, of science is based on entities comprised of matter and energy, which is called natural phenomena in the definition presented above. My definition is detailed and descriptive, and it is presented in the following two paragraphs.

Based on the way I am using the terminology, science is the systematic **study** of the structure, behavior, and properties of entities that are comprised of **matter and/or energy**. The **study** is based on observations and experimentation, often to verify or refute a hypothesis. The entities that are studied in science **have a physical existence**, **independent of the observer**, and they spontaneously **INTERACT** with the environment and other entities. At the simplest level, this involves changes in temperature of the entity, and/or absorption of energy that relates to the entity.

The entities that are studied in science **ALWAYS** have dynamic properties, they interact with the environment, and/or other entities, and they ultimately undergo spontaneous changes, over time, without human intervention.

For example, a boulder (a large rock) is comprised of molecules that are vibrating, and it expands and contracts slightly as the temperature changes. Over an extended period of time, the structure of the boulder will change. A few of the factors that cause a change in structure with this example, are erosion, reactions with chemicals in the environment, freezing of water in between the cracks of the boulder, movements and collisions with the boulder, such as from landslides, or earthquakes.

Mathematics is Not a Science, Based on the Definitions Presented in the Previous Paragraphs

According to the definitions presented above, science is based on experimental evidence. However, mathematics is primarily based on logical proofs, which consist of deductive reasoning. Science can also be based on logical proof of a concept, but in science, the logical proof must be experimentally verified.

Experimentation can also be used to provide evidence of the validity of a mathematical concept. However, the experimental evidence by itself is **not** considered adequate by mathematicians, and to establish validity of a concept, a logical proof based on

deductive reasoning is required. **The above is one of the major differences between mathematics and science.**

Another major difference between mathematics and science, involves the structure of the entities that are studied. In science the entities that are studied are comprised of matter and/or energy. In mathematics, the entities that are studied are not comprised of matter or energy, and they usually consist of symbols, numbers, definitions, postulates, theorems, and/or geometric structures. The geometric structures are often idealized concepts that do not exist in nature. Most, if not all of the entities studied in mathematics are conceptualizations that only exist in the human mind, on paper, or on a computerized display. Some obvious examples are one-dimensional lines, points with no dimensions, and two-dimensional geometric structures, such as perfect, circles, squares, triangles, and rectangles.

What Are the Fundamental Components that Comprise Mathematics

The fundamental components of mathematics are definitions, which only have meaning to human beings. Some of the definitions represent entities that only exist in the human mind, or on a computer screen. Some examples, of the most important definitions in mathematics are numbers, circles, spheres, angles, lines, points, triangles, sets, sequences, equations, and

inequalities. With the basic definitions of mathematics, postulates can be created, which can be used to create theorems. With the definitions, postulates, and/or theorems, mathematical techniques can be created based on logic, such as addition, subtraction, multiplication, division, how to solve algebraic equations and inequalities.

The simplest example of definitions used to create postulates and theorems are numbers. The quantities of solid objects are apparent in nature to human beings, and some animals. Giving each quantity a name, such as one, two, three, is the process of defining specific quantities, which is probably how mathematics started. With the definitions of quantities, in terms of numbers, humans PROBABLY devised counting, addition, subtraction, multiplication, and division.

The important idea to understand is if we start with a different set of definitions, we could obtain a different set of postulates and theorems, and perhaps a different set of mathematical techniques. This is because the postulates of mathematics represent the PROPERTIES of the **concepts that are defined**. (This concept applies to any type of deductive reasoning, not just mathematics.)

To illustrate the above I will use **three definitions of a triangle**, the first is based on the conventional Euclidean geometry, and two are based on Non-Euclidean geometry. Specifically, the second definition is based on spherical geometry,

and the third on hyperbolic geometry. With each of these definitions, the properties and related postulates of a triangle will change.

Definition 1) If we define a triangle as a three-sided polygon, on a flat plane, it is possible to prove that the sum of the three angles of a triangle is always 180°. This is of course based on the conventional Euclidean geometry.

Definition 2) However, if we define a triangle as a three-sided polygon, on the surface of a sphere, we have a triangle with different properties. This triangle essentially is comprised of three line segments that curve outward. Left click here to see an example of this type of triangle. With this triangle, the sum of the angles will be greater than 180°.

Definition 3) On a sphere, the line segments comprising a triangle are curved outward as explained above. We can also define a triangle based on the opposite of the above, which is a triangle comprised of three line segments that are curved inward. In this case, the sum of the three angles would be less than 180°. Left click on the blue underlined words to see examples of this type of triangle, and for details left click on: hyperbolic triangles, and Angle Sum of Triangles in Hyperbolic Geometry.

The point here is the definitions determine the validity of the postulates, that comprise mathematics, and the postulates in turn determine the validity of the theorems. This concept also applies to other types of reasoning. Often people come to different conclusions about the same phenomena, because they are starting with a different set of fundamental definitions. The differences in definitions might be based on basic values and/or goals of the individuals. However, in general, all deductive reasoning starts with basic definitions, which ultimately determine the validity of the conclusions.

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